① (i) Note that A is a $n\times n$ matrix with $A_{ij} = \begin{cases} 1, & (v_i, v_j)\in E \\ 0, & \text{else}. \end{cases}$

We prove by induction on l.

$l=1$ clear ✓

The number of paths of length $(l+1)$ from v_i to v_j is

$$\sum_{a\in N(v_j)} \#\text{paths length } l \text{ from } v_i \text{ to } a$$

$$= \sum_{k=1}^{n} (A^l)_{ik}(A)_{kj} = (A^{l+1})_{ij}$$

by induction

(ii) Suppose $(-d)$ is an eigenvalue with $\varphi\neq 0$ evector. Since A real, symmetric, then $\varphi\in\mathbb{R}^n$ (real vector)

Then $d\|\varphi\|^2 = \langle d\varphi,\varphi\rangle = -\langle A\varphi,\varphi\rangle$

$$= \sum_{(v,w)\in E} (-\varphi(v))\varphi(w)$$

$$\leq \frac{1}{2}\sum_{(v,w)\in E} |\varphi(v)|^2 + |\varphi(w)|^2$$

(using that $ab \leq \frac{|a|^2+|b|^2}{2}$)

$$= d\|\varphi\|^2.$$

Hence we must have equality everywhere, which implies $\varphi(v) = -\varphi(w)$ whenever $(v,w) \in E$.

This concludes if G is connected, take $V = V^+ \cup V^-$, where $V^+ = \{v : \varphi(v) > 0\}$
$V^- = \{v : \varphi(v) < 0\}$.

Otherwise, look at restriction on each connected component.

Conversely, if G bipartite with $E \subset V_1 \times V_2$, where $V = V_1 \sqcup V_2$,
then $\varphi(v) = \begin{cases} 1, & v \in V_1 \\ -1, & v \in V_2 \end{cases}$ is eigenvector with eigenvalue $-d$.

(iii) Note that if G connected, same argument as above holds that φ eigenvector for eigenvalue d, then $\varphi(v) = \varphi(w)$, $\forall (v,w) \in E$
$\Rightarrow \varphi =$ constant (non-zero)
$\Rightarrow$ multiplicity 1.

Now assume G has k connected components. Then up to rearranging the vertices V, we can write A as block matrix

$$A = \begin{pmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ 0 & & & \boxed{A_k} \end{pmatrix}, \quad V = \bigsqcup_{j=1}^{k} V_k$$

$(v,w) \in E \Rightarrow v, w \in V_j$, for some j.

We can consider A_j acts on each $l^2(V_j)$.

Take $\varphi_j(v) = \begin{cases} 1, & v \in V_j \\ 0, & \text{else}. \end{cases}$

Then we can see directly that the eigenspace of evalue d is generated by $\varphi_1, \dots, \varphi_k$, each evector can be written uniquely as

$$\varphi = \sum_{j=1}^{k} \alpha_j \varphi_j, \quad \alpha_j \in \mathbb{R}.$$

② Let $I \subset V$ an independent set (there is no $(v,w) \in E$ with $v, w \in I$).

Let $1_I(v) = \begin{cases} 1, & v \in V \\ 0, & \text{else} \end{cases}$

Then $\langle 1_I, M 1_I \rangle = 0$ by definition.

Clearly $\langle 1_I, 1 \rangle = |I|$ and $\|1_I\|_2 = |I|^{1/2}$.

Direct application of 7.7 gives $|I| \leq (1-\varepsilon) n$.

③ Let $V = \bigsqcup_{j=1}^{k} V_j$ such that each V_i is an independent set. Then from previous exercise that $|V_i| \leq (1-\varepsilon)n$, for all $1 \leq i \leq k$.

But since $\sum_{i=1}^{k} |V_i| = n$, it implies $k \geq (1-\varepsilon)^{-1}$,

hence $\chi(G) \geq (1-\varepsilon)^{-1}$.

④ (i) Apply 7.7 with $f_1 = \delta_w$, $f_2 = \delta_v$.

We have $|P_{v,w}^{l}| = d^{l} \cdot (M^{l})_{v,w}$
$= \langle \delta_w, M^{l} \delta_v \rangle \cdot d^{l}$.

Conclusion follows.

(ii) Note that $l \geq \mathrm{diam}(G)$, then $\forall v, w \in V$, $\exists\, k \leq l$ such that $|P_{v,w}^{k}| \geq 0$.

Using (i), it suffices to show for $C > 0$ suff large, we have $\frac{1}{n} \geq C \cdot (1-\varepsilon)^{l}$

$$\Leftrightarrow l \cdot \log\left(\frac{1}{1-\varepsilon}\right) \geq \log C + \log n$$

$$\Leftrightarrow l \geq \frac{\log C}{\log\left(\frac{1}{1-\varepsilon}\right)} + \frac{\log n}{\log\left(\frac{1}{1-\varepsilon}\right)} \geq \log C + \frac{\log n}{\log\left(\frac{1}{1-\varepsilon}\right)}.$$

(iii) Look at Thm 7.8 in the notes

(5) To show that $\vartheta^{(\ell)} \xrightarrow{w^*} \mu$, it suffices to show that for all $f \in L^2(V)$, we have that $\vartheta^{(\ell)}(f) \to \mu(f)$ as $\ell \to \infty$.

But $\vartheta^{(\ell)}(f) = \vartheta_0(M^\ell f) = \sum_{v \in V} (M^\ell f)(v)\, \vartheta_0(v)$

$$= \langle \vartheta_0, M^\ell f \rangle$$

$$= \frac{\langle \vartheta_0, 1 \rangle \langle f, 1 \rangle}{n} + O\left((1-\varepsilon)^\ell \|\vartheta_0\| \|f\|\right)$$

$$= \mu(f) + O_{\vartheta_0, f}\left((1-\varepsilon)^\ell\right).$$

Conclusion follows.